CLUSTER EXPANSIONS AND THE THEORY OF MANY-BOSON SYSTEMS

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Cluster Expansions and the Theory of Many-Boson Systems[*]

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The previous configuration-space cluster integral treatment of the properties of the ground state of a many-boson system is modified by including those higher order diagrams consistent with the pair excitation approximation, which were previously omitted. The cluster expansions involve a parameter, analogous to the fugacity in classical expansions, whose definition automatically accounts for the depletion of the free particle ground state. The resulting expectation value for the Hamiltonian is in agreement with that obtained by field-theoretic methods for states of pair-excitation type.

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I.

The properties of a many-boson system of particles with repulsive interactions have been treated previously by using perturbation theory in the second quantization formalism[1,2,3] as well as by considering the wave function in configuration space using the theory of cluster expansions.[4] In these treatments certain higher order contributions to the energy, consistent with the pair excitation approximation were omitted. The purpose of the present discussion is to modify the latter configuration space treatment by including previously omitted diagrams and thus obtain complete expressions for the expectation value of the Hamiltonian and other quantities characteristic of the ground state in the pair approximation. The present cluster expansions involve a parameter, analogous to the fugacity in classical expansions, whose definition automatically accounts for the depletion of the free particle ground state.

II.

To do the above, we recall that the pair approximation yields a ground state which in configuration space has the form [see Eq. (A.21) of Appendix II of reference 3]

$$(1) \qquad \psi = {\prod_{i<j=1}^{N}}' [1 + f(r_{ij})]$$

where the prime denotes that in the expanded product for ψ all terms with repeated particle indices are omitted.

Evaluation of the ground state energy is made by variation of $\langle H \rangle$, the expectation value for the Hamiltonian of the Bose system of N interacting particles, given by

$$(2)\qquad \langle H\rangle \int \psi^* \psi \, d\underline{r}^N = \int \psi^* \left[\frac{-h^2}{2m} \sum_{i=1}^{N} \nabla_i^2 + \sum_{i<j=1}^{N}\sum V(r_{ij})\right] \psi \, d\underline{r}^N$$

where $d\underline{r}^N = d\underline{r}_1 \; d\underline{r}_2 \; \ldots \; d\underline{r}_N$ and $V(r_{ij})$ is the two-body interaction. For the ground state wave function ψ given in the form of equation (1), it has previously been shown that this multidimensional integral in (2) may be reduced to an integration over the relative distance between any pair of particles. The resulting expression for the expectation value of the potential energy per particle is[5]

$$(3)\qquad \frac{\langle V\rangle}{N} = \frac{\rho}{2} \int V(r_{12}) \, D(r_{12}) \, d\underline{r}_{12} \,.$$

$D(r)$ is related to the pair distribution function $n_2(r_{12})$ by

$$(4)\qquad D(r) = \frac{n_2(r_{12})}{\rho^2} \,.$$

To facilitate the diagrammatic analysis of the pair distribution function, $D(r_{12})$ is replaced by

$$(4')\qquad D(r) = C_{V_1}(r_{12}) + 2f(r_{12}) \, C_{V_2}(r_{12}) + f^2(r_{12}) \, C_{V_3}(r_{12}) \,.$$

In view of Eq. (1),

$$(5)\qquad n_2(r_{12}) = N(N-1) \frac{\int [\prod{}' [1 + f(r_{ij})]^2 \, d\underline{r}_3 \; \ldots \; d\underline{r}_N}{\int [\prod{}' [1 + f(r_{ij})]^2 \, d\underline{r}^N} \,.$$

We expand these C_V functions in cluster expansions of the following form

(6) $$C_{v_i}(r_{12}) = \frac{z^2}{2} \sum_{n=o}^{\infty} z^n \, b_{n_i}(r_{12})$$

where

(7) $$z = N \frac{\int \psi^2(r_1, \ldots, r_{N-1}) \, d\underline{r}^{N-1}}{\int \psi^2(r_1, \ldots, r_N) \, d\underline{r}^N}$$

and

(8) $$b_{n_i}(r_{12}) = \frac{1}{n!} \int \sum_{(i)} \prod f(r_{ij}) \, d\underline{r}_3 \ \ldots \ d\underline{r}_{n+2} \, .$$

As discussed in the Appendix, the quantity z automatically allows the depletion of the ground state. The integrand $\sum_{(i)} \prod f(r_{ij})$ indicates the sum of all those cluster diagrams for which each of the n particles of the set n is at least singly connected to either or both of the pair 1,2. Only those connected products are allowed which are consistent with the assumption of nonrepeated indices in the wave function ψ. This constraint has caused the cluster diagrams to involve $f(r_{ij})$ instead of $2f(r_{ij})$, as one would expect in the formally similar expansions in classical statistical mechanics.

The expansion parameter z, analogous to the classical fugacity, from its definition, is determined by the following constraint on the total number of particles, which follows directly from (7):

(9) $$\rho = \sum_{\ell=o}^{\infty} \ell \, B_\ell \, z^\ell$$

where

$$B_\ell = \frac{1}{\ell! V} \sum^{(\ell)} \int \prod f(r_{ij}) d\underline{r}_1, \ \ldots, \ d\underline{r}_\ell$$

and $\sum^{(\ell)}$ indicates symbolically summation over all the possible single cluster diagrams which can be formed from ℓ given particles. Here again only those cluster diagrams are allowed which are consistent with nonrepeated indices. Application of this constraint as well as the fact that $f(r_{12})$ has no zero momentum components, simplifies the prescription for the cluster expansion formalism.

The prescription for drawing diagrams for the cluster expansion determining z may be stated as:

For the term for ℓ particles, $\ell = 1,2,4,6, \ldots$.

Connect particle 1 to particle 2, 2 to 3, 3 to 4, ..., ℓ-1 to 1. The diagrams are illustrated in Figure 1. Nonrepeated indices have eliminated the diagrams for $\ell = 3,5, \ldots$. We use this diagrammatic prescription to evaluate the constraining equation (9).

Now

$$B_1 = 1$$

and

$$B_2 = \frac{1}{2!} \frac{1}{V} \sum^{(2)} \int f^2(r_{12}) d\underline{r}_{12} = \frac{1}{2} \int f^2(r_{12}) d\underline{r}_{12}$$

while

$$B_n = 0, \qquad n = 3,5, \ldots$$

$$B_n = \frac{1}{n! V} \sum^{(n)} \int \prod f(r_{ij}) d\underline{r}_1 \ \ldots \ d\underline{r}_n, \qquad n = 4,6, \ldots .$$

To evaluate B_n, we note that permuting the $n - 1$ particles other than 1 produces $1/2(n - 1)!$ configurations, while there are two ways of drawing the configuration from ψ and ψ^*. Thus

$$B_n = \frac{1}{nV} \int f(r_{12})\, f(r_{23}) \;\ldots\; f(r_{n,1})\, d\underline{r}_1 \ldots d\underline{r}_n, \quad n \text{ even.}$$

The introduction of $\gamma(k)$, the Fourier transform of $f(r)$,

$$(10) \qquad \gamma(k) = \int f(r)\, e^{-i\underline{k}\cdot\underline{r}}\, d\underline{r}$$

into this expression for B_n yields

$$(11) \qquad B_n = \frac{1}{(2\pi)^3 n} \int [\gamma(k)]^n\, d\underline{k}\, .$$

Using this in Eq. (9), we have

$$(12) \qquad \rho = z + \frac{1}{(2\pi)^3} \int \frac{z^2\gamma^2}{1 - z^2\gamma^2}\, d\underline{k}\, .$$

Thus z is expressed in terms of ρ and γ and may be determined by successive approximation of this relation.

Consider now the third term in $\langle V\rangle/N$,

$$V(r_{12})\, f^2(r_{12})\, C_{V_3}(r_{12})\, .$$

The constraint of nonrepeated indices requires $C_{V_3} = 1$, inasmuch as one $f(r_{12})$ must come from ψ and the other from ψ^*, then products connecting particles one and two are impossible. For the purposes of the eventual evaluation and summation, this diagram is entered in the first position of Figure 4.

The second term in $\langle V\rangle/N$ is

$$V(r_{12})\, f(r_{12})\, C_{V_2}(r_{12})\, .$$

The expansion of the type of Eq. (6) is

$$C_{v_2}(r_{12}) = \frac{z^2}{\rho^2}\left[1 + \sum_{n=1}^{\infty} z^n\, b_{n_2}(r_{12})\right] \tag{13}$$

where

$$b_{n_2}(r_{12}) = \frac{1}{n!}\int \sum_{(2)} \prod f(r_{ij})\, d\underline{r}_3 \cdots d\underline{r}_{n+2}\,.$$

To evaluate $\sum_{(2)} \prod f(r_{ij})$ we note that terms of the type $V(r_{12})\ f(r_{12}) \times \prod f(r_{ij})$ are involved. Certain connected products are eliminated as illustrated in Figure 2. The terms which remain to be considered are represented by ring-type diagrams (see Figure 3).

Nonrepeated indices require that for these diagrams

$$\sum_{(2)} \prod f(r_{ij}) = 0, \qquad n \ \text{odd}$$

$$\sum_{(2)} \prod f(r_{ij}) = (n)!\ f(r_{13})\, f(r_{34}) \cdots f(r_{n+2,2})\,.$$

Thus

$$b_{n_2}(r_{12}) = 0, \qquad n \ \text{odd}$$

$$= \frac{1}{(2\pi)^3}\int [\gamma(k)]^{n+1}\, e^{i\underline{k}\cdot\underline{r}}\, d\underline{k}, \qquad n \ \text{even}\,.$$

This group of diagrams (n even) are entered in the remaining places of the first row and column of Figure 4 for their eventual summation. The contribution from $n = 0$ to $\langle V\rangle/N$ is

(15) $$\frac{1}{(2\pi)^3\rho}\int z\,\nu(k)\; z\gamma(k)\; d\underline{k}$$

where

(16) $$\nu(k) = \int V(r)\; e^{i\underline{k}\cdot\underline{r}}\; d\underline{r}$$

The first term in $\langle V\rangle/N$ is

$$V(r_{12})\; C_{v_1}(r_{12}) \;.$$

$C_{v_1}(r_{12})$ is also expanded in a cluster expansion of the form of Eq. (6) - (8)

(17) $$C_{v_1}(r_{12}) = \frac{z^2}{\rho^2}\sum_{n=0}^{\infty} z^n\; b_{n_1}(r_{12}) \;,$$

where

$$b_{n_1}(r_{12}) = \frac{1}{n!}\int \sum_{(1)}\prod f(r_{ij})\; d\underline{r}_3 \;\ldots\; d\underline{r}_{n+2} \;.$$

Of these cluster diagrams included in the sum $\sum\limits_{(1)}\prod f(r_{ij})$ by the definition following Eq. (8) certain diagrams are not allowed because $f(r_{ij})$ does not have zero momentum components or because of the hypothesis of nonrepeated indices.

If more than two f's emanate from a single particle, existence of repeated indices is implied in ψ. Those cluster diagrams which do contribute to $V(r_{12})\; C_{v_1}(r_{12})$ are illustrated for $n = 0, 1, \ldots$ in Figure 5. The diagrams have been grouped into three columns for this discussion.

Consider first the diagrams in the left or first column. Call the contribution of a diagram in the n^{th} row of this column to $b_{n_1}(r_{12})$, $b^{(1)}_{n_1}(r_{12})$.

Then

$$b_{n_1}^{(1)}(r_{12}) = \frac{1}{n!}\sum_{(1)}\prod^{(1)} \int f(r_{13})\, f(r_{34}) \;\ldots\; f(r_{n+2,2})\, d\underline{r}_3 \ldots d\underline{r}_{n+2} .$$

Permutation of the n particles produces $n!$ similar configurations while there are also two ways of drawing each configuration from ψ and ψ^*. Hence

$$b_{n_1}^{(1)}(r_{12}) = 2\int f(r_{13}) \;\ldots\; f(r_{n+2,2})\, d\underline{r}_3 \ldots d\underline{r}_{n+2} .$$

Introduction of $\gamma(k)$, the Fourier transform of $f(r)$, (Eq. 10) and summing up the $b_{n_1}^{(1)}(r_{12})$ yields for the contribution of the first column to $C_{V_1}(r_{12})$

$$(18) \qquad \frac{z^2}{\rho^2}\sum_{n=1}^{\infty} z^n\, b_{n_1}^{(1)}(r_{12}) = \frac{z^2}{\rho^2}\,\frac{2}{(2\pi)^3 z}\int \frac{z^2\gamma^2(k)}{1 - z\gamma(k)}\, e^{i\underline{k}\cdot\underline{r}_{12}}\, d\underline{k}$$

for $|z\gamma(k)| < 1$. The contribution to $\langle V\rangle/N$ is

$$(19) \qquad \frac{1}{(2\pi)^3\rho}\int z\nu(k)\,\frac{z^2\gamma^2(k)}{1 - z\gamma(k)}\, d\underline{k} .$$

Consider next the diagrams in the third column and call them $b_{n_1}^{(3)}(r_{12})$. Examine the contribution of a typical one, say $b_{4_1}^{(3)}(r_{12})$ to $\langle V\rangle/N$:

$$(20) \qquad \frac{\rho}{2}\int \frac{z^2}{\rho^2} V(r_{12})[z^4\, b_4^{(3)}(r_{12})]\, d\underline{r}_{12}$$

$$= \frac{\rho}{2}\,\frac{z^2}{\rho^2}\,\frac{1}{4!}\,\frac{4!}{2}\, 2\int V(r_{12})\, f(r_{23})\, f(r_{23})$$

$$\times\; f(r_{14})\, f(r_{45})\, f(r_{56})\, f(r_{61})\, d\underline{r}_3\, d\underline{r}_4\, d\underline{r}_5\, d\underline{r}_6\, d\underline{r}_{12}$$

$$= \frac{\rho}{2}\,\frac{z^2}{\rho^2}\,\nu(0)\,\frac{1}{(2\pi)^6}\int z\gamma^2(k)\, d\underline{k}\cdot\int z^3\,\gamma^4(k')\, d\underline{k}' .$$

Summing up all these diagrams, and noting Eq. (12), we obtain for $|z\,\gamma(k)| < 1$

$$\frac{\langle V\rangle^{(3)}}{N} = \frac{\rho\, v(0)}{2} . \tag{21}$$

The diagrams represented in the second column remain to be considered. These connected products are of the "separable" type. They separate into the product of the connected product for particles one and two and the particles above them in the diagram, and the connected product for particles one and two and those particles below them in the diagram. In general, there will be no connected products with an even number of particles above an imaginary line joining one and two and an odd number below, or vice-versa. Thus for n odd, there are no separable connected products.

These connected products for n even are entered in Figure 6 and the remaining places of Figure 4.

We may use these arrays of diagrams to assist in the summation of their contribution to $\langle V\rangle/N$. First consider Figure 4. Summing the contribution of the first row to $\langle V\rangle/N$ and transforming to momentum space, one has

$$\frac{z^2}{2\rho}\,\frac{1}{(2\pi)^6}\int v(\underline{k}-\underline{k}')\;\gamma(k)\;\frac{\gamma(k')}{1-z^2\gamma^2(k')}\;d\underline{k}\;d\underline{k}' .$$

Similarly the second row contributes

$$\frac{z^2}{2\rho}\,\frac{1}{(2\pi)^6}\int v(\underline{k}-\underline{k}')\;z^2\gamma^3(k)\;\frac{\gamma(k')}{1-z^2\gamma^2(k')}\;d\underline{k}\;d\underline{k}' .$$

Summing over the contribution of all the rows,

$$\frac{1}{(2\pi)^6}\,\frac{z^2}{2\rho}\int \nu(\underline{k}-\underline{k}')\,\{\gamma(k)+z^2\gamma^3(k)+\cdots\}\,\frac{\gamma(k')}{1-z^2\gamma^2(k')}\,d\underline{k}\,d\underline{k}',$$

or

(22) $$\frac{1}{(2\pi)^6}\,\frac{z^2}{2\rho}\int\left[\int \nu(\underline{k}-\underline{k}')\,\frac{\gamma(k')}{1-z^2\gamma^2(k')}\,d\underline{k}'\right]\frac{\gamma(k)}{1-z^2\gamma^2(k)}\,d\underline{k}\;,$$

for $|z\,\gamma(k)| < 1$.

Application of a similar procedure yields the following contribution to $\langle V\rangle/N$ from Figure 6

(23) $$\frac{z^2}{2\rho}\,\frac{1}{(2\pi)^6}\int \nu(\underline{k}-\underline{k}')\left[z\,\gamma^2(k)+z^3\gamma^4(k)+\cdots\right]\frac{z\gamma^2(k')}{1-z^2\gamma^2(k')}\,d\underline{k}'\,d\underline{k}$$
$$=\frac{1}{(2\pi)^6}\,\frac{z^2}{2\rho}\int\left[\int \nu(\underline{k}-\underline{k}')\,\frac{z\gamma^2(k')}{1-z^2\gamma^2(k')}\,d\underline{k}'\right]\frac{z\gamma^2(k)}{1-z^2\gamma^2(k)}\,d\underline{k}\,.$$

The sum of Equations (15), (21), (22), and (23) is the complete $\langle V\rangle/N$, i.e.,

(24) $$\frac{\langle V\rangle}{N}=\frac{1}{2}\,\rho\,\nu(0)+\frac{1}{(2\pi)^3\rho}\int z\,\nu(k)\,\frac{z\gamma(k)}{1-z\,\gamma(k)}\,d\underline{k}$$
$$+\frac{1}{(2\pi)^3\rho}\int \frac{1}{2}\,I_2(k)\,\frac{z^2\gamma^2(k)}{1-z^2\gamma^2(k)}\,d\underline{k}$$
$$+\frac{1}{(2\pi)^3\rho}\int \frac{1}{2}\,I_1(k)\,\frac{z\,\gamma(k)}{1-z^2\gamma^2(k)}\,d\underline{k}\,,$$

where

(25) $$I_1(k')=-\frac{1}{(2\pi)^3}\int \nu(\underline{k}-\underline{k}')\,\frac{z\,\gamma(k)}{1-z^2\gamma^2(k)}\,d\underline{k}\;,$$

(26) $$I_2(k')=\frac{1}{(2\pi)^3}\int \nu(\underline{k}-\underline{k}')\,\frac{z^2\gamma^2(k)}{1-z^2\gamma^2(k)}\,d\underline{k}\;.$$

In a similar manner to this treatment of $\langle V\rangle/N$, use of cluster expansions in $\langle T\rangle/N$ yields

$$\frac{\langle T\rangle}{N} = -\frac{\hbar^2}{2m}\frac{1}{(2\pi)^3\rho}\int k^2 \frac{z^2\gamma^2(k)}{1 - z^2\gamma^2(k)}\, d\underline{k}\,. \tag{27}$$

The expectation value for the energy per particle is the sum of Eq. 24 and Eq. 27. This sum is equivalent to that of Girardeau and Arnowitt,[6] obtained by field-theoretic methods for states of pair-excitation type.(*) The quantity $-z\,\gamma(k)$ should be identified with their $\varphi(k)$ and z with ρ_o.

Varying $\frac{\langle H\rangle}{N}$ with respect to $\gamma(k)$ yields the Euler equation

$$F(\gamma,\, v)\Big\{[z\,v(k) - I_1(k)][1 + z^2\gamma^2(k)] + 2\Big[\frac{\hbar^2}{2m}k^2 + z\,v(k) + I_2(k) + I_1(0) - I_2(0)\Big]\, z\,\gamma(k)\Big\} = 0, \tag{28}$$

where F does not vanish because of the boundary conditions and $|z\,\gamma(k) < 1$. This equation may be solved by iteration.

Substitution of the solution of this Euler equation or the minimizing $\gamma(k)$ into the expression for $\langle H\rangle/N$ leads to the following minimal form:

$$\frac{\langle H\rangle}{N}\bigg|_{min} = \frac{1}{2}\rho\, v(0) + \frac{1}{(2\pi)^3\rho}\int \frac{z\,v(k) - I_1(k)}{2}\, z\,\gamma(k)\, d\underline{k} - \frac{1}{(2\pi)^3\rho}\int\Big[\frac{1}{2} I_2(k) + I_1(0) - I_2(0)\Big]\frac{z^2\gamma^2(k)}{1 - z^2\gamma^2(k)}\, d\underline{k}\,. \tag{29}$$

One may utilize these expressions to obtain asymptotic expressions for the ground state energy. Recalling the meaning of the pair excitation states as eigenstates of either the dilute, strongly-coupled system

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(*) The equivalent of the absence of zero momentum components of $f(r)$ is implied in their treatment, when the transformation $\sum_k' \to V/(2\pi)^3 \int d\underline{k}$ is made.

or in the limit of weak-coupling, one may consider either a low-density or a weak-coupling expansion of the ground-state energy for the pair-excitation states. The latter was considered by Girardeau[7] and the previous calculation[4] was based on the former. To the order of that calculation z becomes just ρ and the I_1 and I_2 terms do not contribute. In either case, use of the pseudopotential of Lee, Huang, and Yang[3]

(30) $$V(r) = 8\pi a \frac{\hbar^2}{2m} \delta(r) \frac{\partial}{\partial r} (\quad)$$

leads to

(31) $$4\pi \rho a \frac{\hbar^2}{2m} \left[1 + \frac{128}{15\sqrt{\pi}} (\rho a^3)^{\frac{1}{2}}\right]$$

for the first two terms in the asymptotic expression for the ground state energy.[4][7]

The pair distribution function, obtained by summing up the cluster expansions, is

(32) $$D(\underline{r}) = 1 + \frac{2z}{\rho} J_3(\underline{r}) + J_1^2(\underline{r}) + J_2^2(\underline{r}) ,$$

where

$$J_1(\underline{r}) = -\frac{1}{(2\pi)^3\rho} \int \frac{z\,\gamma(k)}{1 - z^2\gamma^2(k)} e^{i\underline{k}\cdot\underline{r}} d\underline{k} ,$$

$$J_2(\underline{r}) = \frac{1}{(2\pi)^3\rho} \int \frac{z^2\gamma^2(k)}{1 - z^2\gamma^2(k)} e^{i\underline{k}\cdot\underline{r}} d\underline{k} ,$$

$$J_3(\underline{r}) = \frac{1}{(2\pi)^3\rho} \int \frac{z\,\gamma(k)}{1 - z\;\gamma(k)} e^{i\underline{k}\cdot\underline{r}} d\underline{k} .$$

As noted by Girardeau and Arnowitt,[6] this expression for the pair distribution function has rather unrealistic behavior close to the origin. This is especially true when considering a system with strong repulsive cores. Such a failing is not surprising when proceeding in configuration space, since the wave function employed [Eq. (1)] can vanish only approximately inside the core and hence the pair distribution function constructed from it cannot be expected to do better. In this regard, is should be added that the use of this wave function was motivated by the pseudopotential method — a method whose eventual object is to obtain an extrapolated wave function which is valid outside the core, the real wave function vanishing inside the core. Thus the pair distribution function should vanish inside the core, this calculated one vanishing to the proper order at the core.

III.

It is now evident that the cluster expansion formalism in configuration space and the formalism of second quantization in momentum space through the equivalent approximation lead to the same results. The reason underlying this equivalence is of primary importance. In the second quantization formalism, the restriction to pair excitation states reduces the Hamiltonian operator involved from a complicated quadri-linear form (in the plane wave creation and destruction operators) to a simple bilinear form. This reduction permits diagonalization of the Hamiltonian by a canonical transformation. In the equivalent approximation in configuration space, the restriction to nonrepeated indices simplifies the quite intractible expression for the pair distribution function, by selecting out of the original hierarchy of admissible diagrams, only certain diagrams of a ring nature. The simple structure of these diagrams enables one to evaluate exactly their contribution to the pair distribution function. This fact utilized previously, for example, in the Debye-Hückel theory of electrolytes[8][9], in the Kahn-Uhlenbeck treatment of the perfect Bose-Einstein gas[10][9], and in the Nakamura treatment of superconductivity[11], has again formed the basis for the present calculation. In contrast with several cluster treatments of the Boson systems[5,12,13] the selection of ring diagrams is not made arbitrarily, i.e., because their contribution may be explicitly summed, but is a necessary consequence of the original hypothesis.

APPENDIX

We show in this appendix that

$$vz = \frac{V \int \psi^2(\underline{x}_1, \ldots, \underline{x}_{N-1})\, d\underline{x}^{N-1}}{\int \psi^2(\underline{x}_1 \ldots, \underline{x}_N)\, d\underline{x}^N} \tag{A1}$$

in this approximation measures the fraction in the ground state.

Now $n_{\underline{k}_1}$, the number of particles having momentum k_1 is

$$\frac{N \int \psi^2(\underline{k}_1, \ldots, \underline{k}_N)\, d\underline{k}_2 \ldots d\underline{k}_N}{\int \psi^2(\underline{k}_1, \ldots, \underline{k}_N)\, d\underline{k}_1 \ldots d\underline{k}_N}$$

where $\psi(\underline{k}_1, \ldots, \underline{k}_N)$ is the momentum probability amplitude.

But

$$\psi(\underline{k}_1, \ldots, \underline{k}_N) = \int \psi(\underline{x}_1, \ldots, \underline{x}_N)\, e^{i(\underline{k}_1 \cdot \underline{x}_1 + \cdots + \underline{k}_N \cdot \underline{x}_N)}\, d\underline{x}^N$$

and

$$\psi^2(\underline{k}_1, \ldots, \underline{k}_N) = \int \psi(\underline{x}'^N)\, \psi(\underline{x}^N)\, e^{i[\underline{k}_1 \cdot (\underline{x}_1 - \underline{x}_1') + \cdots + \underline{k}_N \cdot (\underline{x}_N - \underline{x}_N')]}$$

$$d\underline{x}_1\, d\underline{x}_1' \ldots d\underline{x}_N\, d\underline{x}_N' .$$

Therefore

$$n_{\underline{k}_1} = \frac{N \int \psi(\underline{x}_1', \underline{x}_2, \ldots, \underline{x}_N)\, \psi(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_N)\, e^{i[\underline{k}_1 \cdot (\underline{x}_1 - \underline{x}_1')]}\, d\underline{x}_1'\, d\underline{x}^N}{V \int \psi^2(\underline{x}_1, \ldots, \underline{x}_N)\, d\underline{x}^N} \tag{A2}$$

and

$$n_o = \frac{N \int \psi^*(\underline{x}_1', \underline{x}_2, \ldots, \underline{x}_N)\, \psi(\underline{x}_1, \ldots, \underline{x}_N)\, d\underline{x}_1'\, d\underline{x}^N}{V \int \psi^2(\underline{x}_1, \ldots, \underline{x}_N)\, d\underline{x}^N} .$$

For $\psi = \prod'[1 + f(x_{ij})]$, any integral in the numerator of Eq. (A2) which involves $f(x_1' - x_j)$ or $f(x_1 - x_k)$ vanishes. This means that particle one is not present in the numerator and

$$\frac{n_o}{N} = V \frac{\int \psi^2(\underline{x}_2, \ldots, \underline{x}_N)\, d\underline{x}_2 \ \ldots \ d\underline{x}_N}{\int \psi^2(\underline{x}_1, \ldots, \underline{x}_N)\, d\underline{x}_1 \ \ldots \ d\underline{x}_N},$$

i.e.,

$$\frac{n_o}{N}, \text{ the fraction in the ground state} = vz.$$

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